

Stochastic solutions of nonlinear PDE's and an extension of superprocesses

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Abstract

Stochastic solutions provide new rigorous results for nonlinear PDE's and, through its local non-grid nature, are a natural tool for parallel computation. There are two different approaches for the construction of stochastic solutions: MacKean's and superprocesses. However, when restricted to measures, superprocesses can only be used to generate solutions for a limited class of nonlinear PDE's. A new class of superprocesses, namely superprocesses on signed measures and on distributions, is proposed to extend the stochastic solution approach to a wider class of PDE's.

1 Introduction

A stochastic solution of a linear or nonlinear partial differential equation is a stochastic process which, when started from a particular point x in the domain generates after time t a boundary measure which, when integrated over the initial condition at $t = 0$, provides the solution at the point x and time t . For example for the heat equation

$$\partial_t u(t, x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u(t, x) \quad \text{with} \quad u(0, x) = f(x) \quad (1)$$

the stochastic process is Brownian motion and the solution is

$$u(t, x) = \mathbb{E}_x f(X_t) \quad (2)$$

\mathbb{E}_x meaning the expectation value, starting from x , of the process

$$dX_t = dB_t \quad (3)$$

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The domain here is $\mathbb{R} \times [0, t)$ and the expectation value in (2) is indeed the inner product $\langle \mu_t, f \rangle$ of the initial condition f with the measure μ_t generated by the Brownian motion at the t -boundary. The usual integral solution,

$$u(t, x) = \frac{1}{2\sqrt{\pi}} \int \frac{1}{\sqrt{t}} \exp\left(-\frac{(x-y)^2}{4t}\right) f(y) dy \quad (4)$$

with the heat kernel, has exactly the same interpretation. Of course, an important condition for the stochastic process (Brownian motion in this case) to be considered *the* solution of the equation is the fact that the same process works for any initial condition. This should be contrasted with stochastic processes constructed from particular solutions.

That the solutions of linear elliptic and parabolic equations, both with Cauchy and Dirichlet boundary conditions, have a probabilistic interpretation is a classical result and a standard tool in potential theory [1] [2] [3]. In contrast with the linear problems, explicit solutions in terms of elementary functions or integrals for nonlinear partial differential equations are only known in very particular cases. Therefore the construction of solutions through stochastic processes, for nonlinear equations, has become an active field in recent years. The first stochastic solution for a nonlinear pde was constructed by McKean [4] for the KPP equation. Later on, the exit measures provided by diffusion plus branching processes [5] [6] as well as the stochastic representations recently constructed for the Navier-Stokes [7] [8] [9] [10] [11], the Vlasov-Poisson [12] [13] [15], the Euler [14] and a fractional version of the KPP equation [16] define solution-independent processes for which the mean values of some functionals are solutions to these equations. Therefore, they are *exact stochastic solutions*.

In the stochastic solutions one deals with a process that starts from the point where the solution is to be found, a functional being then computed on the boundary or in some cases along the whole sample path. In addition to providing new exact results, the stochastic solutions are also a promising tool for numerical implementation, in particular for parallel computing using the recently developed probabilistic domain decomposition method [17] [18] [19]. This method decomposes the space in subdomains and then uses in each one a deterministic algorithm with Dirichlet boundary conditions, the values on the boundaries being determined by a stochastic algorithm. This minimizes the time-consuming communication problem between domains and allows for extraordinary improvements in computer time.

There are basically two methods to construct stochastic solutions. The first method, which will be called the McKean method, is essentially a probabilistic interpretation of the Picard series. The differential equation is written as an integral equation which is rearranged in a such a way that the coefficients of the successive terms in the Picard iteration obey a normalization condition. The Picard iteration is then interpreted as an evolution and branching process, the stochastic solution being equivalent to importance sampling of the normalized Picard series. The second method constructs the boundary measures of a measure-valued stochastic process (a superprocess) and obtains the solution of

the differential equation by a scaling procedure.

For a detailed comparison of the two methods refer to [20]. Here after a short review of the construction of superprocesses one illustrates the basic reason why this construction, when restricted to measure-valued superprocesses, can only be applied to a limited class of nonlinear partial differential equations. A wider class of superprocesses, namely superprocesses on signed measures and on distributions might provide useful stochastic solutions for some other nonlinear PDE's.

2 The construction of superprocesses

In the past, superprocesses have been constructed in the space $M_+(E)$ of finite measures on a measurable space (E, \mathcal{B}) . Because this setting somehow restricts the range of nonlinear equations for which solutions may be obtained by superprocesses, it is convenient to use a wider framework.

Let \mathcal{S} be the Schwartz space of functions of rapid decrease and $\mathcal{U} \subset \mathcal{S}$ those functions in \mathcal{S} that may be extended into the complex plane as entire functions of rapid decrease on strips. \mathcal{U}' , the dual of \mathcal{U} , is Silva's space of tempered ultradistributions [21] [22], which can also be characterized as the space of all Fourier transforms of distributions of exponential type [23]. Furthermore, for reasons to be clear later on, it is convenient to restrict oneself to the space \mathcal{U}'_0 of tempered ultradistributions of compact support [22].

Denote now by $(X_t, P_{0,\nu})$ a branching stochastic process with values in \mathcal{U}'_0 and transition probability $P_{0,\nu}$ starting from time 0 and $\nu \in \mathcal{U}'_0$. The process is said to satisfy the *branching property* if given $\nu = \nu_1 + \nu_2$

$$P_{0,\nu} = P_{0,\nu_1} * P_{0,\nu_2} \quad (5)$$

that is, after the branching (X_t^1, P_{0,ν_1}) and (X_t^2, P_{0,ν_2}) are independent and $X_t^1 + X_t^2$ has the same law as $(X_t, P_{0,\nu})$. In terms of the *transition operator* V_t operating on functions on \mathcal{U} this is

$$\langle V_t f, \nu_1 + \nu_2 \rangle = \langle V_t f, \nu_1 \rangle + \langle V_t f, \nu_2 \rangle \quad (6)$$

where $e^{-\langle V_t f, \nu \rangle} \doteq P_{0,\nu} e^{-\langle f, X_t \rangle}$ or

$$\langle V_t f, \nu \rangle = -\log P_{0,\nu} e^{-\langle f, X_t \rangle} \quad (7)$$

$f \in \mathcal{U}, \nu \in \mathcal{U}'_0$.

Underlying the usual construction of superprocesses, in the form that is useful for the representation of solutions of PDE's, there is a stochastic process with paths that start from a particular point in E , then propagate and branch but the paths preserve the same nature after the branching. In terms of measures it means that one starts from an initial δ_x which at branching originate other δ'_s with at most some scaling factors. It is to preserve this pointwise interpretation that, in this larger setting, we are restricting to ultradistributions in \mathcal{U}'_0 . Any

ultradistribution in \mathcal{U}'_0 may be represented as a multipole expansion at any point of its support, that is as a series of δ' 's and their derivatives [22]. Therefore any arbitrary transition in the process X_t in \mathcal{U}'_0 may be associated to a branching of paths in E and along these new paths new distributions with point support propagate. As a result the construction now proceeds as in the measure-valued case.

In $M = [0, \infty) \times E$ consider a set $Q \subset M$ and the associated exit process $\xi = (\xi_t, \Pi_{0,x})$ with parameter k defining the lifetime. The process starts from $x \in E$ carrying along an ultradistribution in \mathcal{U}'_0 with support on the path. At each branching point of the ξ_t -process there is a transition ruled by the P probability in \mathcal{U}'_0 leading to one or more elements in \mathcal{U}'_0 . These \mathcal{U}'_0 elements are then carried along by the new paths of the ξ_t -process. The whole process stops at the boundary ∂Q , finally defining a exit process $(X_Q, P_{0,\nu})$ on \mathcal{U}'_0 . If the initial ν is δ_x one writes

$$u(x) = \langle V_Q f, \nu \rangle = -\log P_{0,x} e^{-\langle f, X_Q \rangle} \quad (8)$$

$\langle f, X_Q \rangle$ being computed on the (space-time) boundary with the exit ultradistribution generated by the process.

The connection with nonlinear PDE's is established by defining the whole process to be a (ξ, ψ) -*superprocess* if $u(x)$ satisfies the equation

$$u + G_Q \psi(u) = K_Q f \quad (9)$$

where G_Q is the Green operator,

$$G_Q f(r, x) = \Pi_{0,x} \int_0^\tau f(s, \xi_s) ds \quad (10)$$

and K_Q the Poisson operator

$$K_Q f(x) = \Pi_{0,x} 1_{\tau < \infty} f(\xi_\tau) \quad (11)$$

$\psi(u)$ means $\psi(0, x; u(0, x))$ and τ is the first exit time from Q .

The superprocess is constructed as follows: Let $\varphi(s, x; z)$ be the branching function at time s and point x . Then for $e^{-w(0,x)} \doteq P_{0,x} e^{-\langle f, X_Q \rangle}$ one has

$$P_{0,x} e^{-\langle f, X_Q \rangle} \doteq e^{-w(0,x)} = \Pi_{0,x} \left[e^{-k\tau} e^{-f(\tau, \xi_\tau)} + \int_0^\tau ds k e^{-ks} \varphi(s, \xi_s; e^{-w(\tau-s, \xi_s)}) \right] \quad (12)$$

τ is the first exit time from Q and $f(\tau, \xi_\tau) = \langle f, X_Q \rangle$ is computed with the exit boundary ultradistribution. For measure-valued superprocesses the branching function would be

$$\varphi(s, y; z) = c \sum_{n=0}^{\infty} p_n(s, y) z^n \quad (13)$$

with $\sum_n p_n = 1$ and c denoting the branching intensity, but now it may be a much more general function.

For the interpretation of the superprocesses as generating solutions of PDE's, an essential role is played by a transformation of Eq.(12) that uses $\int_0^\tau k e^{-ks} ds = 1 - e^{-k\tau}$ and the Markov property $\Pi_{0,x} 1_{s < \tau} \Pi_{s,\xi_s} = \Pi_{0,x} 1_{s < \tau}$. This is lemma 1.2 in ch.4 of Ref.[5]. Because it only depends on the Markov properties of the $(\xi_t, \Pi_{0,x})$ process it also holds in this more general context. A proof is included in the Appendix with the notations used in this paper.

Using the lemma, Eq.(12) for $e^{-w(0,x)}$ is converted into

$$e^{-w(0,x)} = \Pi_{0,x} \left[e^{-f(\tau, \xi_\tau)} + k \int_0^\tau ds \left[\varphi \left(s, \xi_s; e^{-w(\tau-s, \xi_s)} \right) - e^{-w(\tau-s, \xi_s)} \right] \right] \quad (14)$$

Eq.(9) is now obtained by a limiting process. Let in (14) replace $w(0,x)$ by $\beta w_\beta(0,x)$ and f by βf . β is interpreted as the mass of the particles and when the \mathcal{U}'_0 -valued process $X_Q \rightarrow \beta X_Q$ then $P_\mu \rightarrow P_{\frac{\mu}{\beta}}$.

$$e^{-\beta w(0,x)} = \Pi_{0,x} \left[e^{-\beta f(\tau, \xi_\tau)} + k_\beta \int_0^\tau ds \left[\varphi_\beta \left(s, \xi_s; e^{-\beta w(\tau-s, \xi_s)} \right) - e^{-\beta w(\tau-s, \xi_s)} \right] \right] \quad (15)$$

Two scaling limits will be used in this paper. The first one, which is the one used in the past for superprocesses on measures, defines

$$u_\beta^{(1)} = (1 - e^{-\beta w_\beta}) / \beta \quad ; \quad f_\beta^{(1)} = (1 - e^{-\beta f}) / \beta \quad (16)$$

and

$$\psi_\beta^{(1)}(0, x; u_\beta^{(1)}) = \frac{k_\beta}{\beta} \left(\varphi \left(0, x; 1 - \beta u_\beta^{(1)} \right) - 1 + \beta u_\beta^{(1)} \right) \quad (17)$$

one obtains from (15)

$$u_\beta^{(1)}(0, x) + \Pi_{0,x} \int_0^\tau ds \psi_\beta^{(1)}(s, \xi_s; u_\beta^{(1)}) = \Pi_{0,x} f_\beta^{(1)}(\tau, \xi_\tau) \quad (18)$$

that is

$$u_\beta^{(1)} + G_Q \psi_\beta^{(1)}(u_\beta^{(1)}) = K_Q f_\beta^{(1)} \quad (19)$$

When $\beta \rightarrow 0$, $f_\beta^{(1)} \rightarrow f$ and if ψ_β goes to a well defined limit ψ then u_β tends to a limit u solution of (9) associated to a superprocess. Also one sees from (16) that in the $\beta \rightarrow 0$ limit

$$u_\beta^{(1)} \rightarrow w_\beta = -\log P_{0,x} e^{-\langle f, X_Q \rangle} \quad (20)$$

as in Eq.(8). The superprocess corresponds to a cloud of particles for which both the mass and the lifetime tend to zero.

3 Measure-valued superprocesses and nonlinear PDE's

Here one restricts oneself to measure-valued superprocesses, that is, in terms of paths, to δ 's propagating along the paths of the $(\xi_t, \Pi_{0,x})$ process and simply branching to new δ measures at each branching point. Let us construct a

superprocess providing a solution to the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - u^\alpha \quad (21)$$

for $1 < \alpha \leq 2$. Comparing with (9) one should have

$$\psi(0, x; u) = u^\alpha$$

Then from (17) and (13), with $z = 1 - \beta u_\beta^{(1)}$ one has

$$\begin{aligned} \varphi(0, x; z) &= \sum_n p_n z^n = z + \frac{\beta}{k_\beta} u_\beta^{(1)\alpha} = z + \frac{\beta}{k_\beta} \frac{(1-z)^\alpha}{\beta^\alpha} \\ &= z + \frac{1}{k_\beta \beta^{\alpha-1}} \left(1 - \alpha z + \frac{\alpha(\alpha-1)}{2} z^2 - \frac{\alpha(\alpha-1)(\alpha-2)}{3!} z^3 + \dots \right) \end{aligned} \quad (22)$$

Choosing $k_\beta = \frac{\alpha}{\beta^{\alpha-1}}$ the terms in z cancel and for $1 < \alpha \leq 2$ the coefficients of all the remaining z powers are positive and may be interpreted as branching probabilities. It would not be so for $\alpha > 2$. Then

$$p_0 = \frac{1}{\alpha}; \quad p_1 = 0; \quad \dots \quad p_n = \frac{(-1)^n}{\alpha} \binom{\alpha}{n} \quad n \geq 2 \quad (23)$$

with $\sum_n p_n = 1$. With this choice of probabilities p_n for branching into new δ measures and with $k_\beta = \frac{\alpha}{\beta^{\alpha-1}}$ and $\beta \rightarrow 0$ one obtains a superprocess which, through (8), provides a solution to the Eq.(21). $\alpha = 2$ is an upper bound for this representation, because for $\alpha > 2$ some of the p'_n s would be negative and would not be interpretable as branching probabilities.

For the particular case

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - u^2 \quad (24)$$

$$p_1 = 0; \quad p_0 = p_2 = \frac{1}{2}; \quad k_\beta = \frac{2}{\beta} \quad (25)$$

When $\beta \rightarrow 0$, the solutions are given by (8) and the superprocesses correspond to the scaling limit of processes where both the mass and the lifetime of the particles tends to zero and at each bifurcation point one has probability p_0 of dying without offspring or creating n new δ measures with probabilities p_n .

Superprocesses are usually associated with nonlinear PDE's in the scaling limit $\beta \rightarrow 0$ of (17)-(18). However other limits may also be useful. For example with $p_n = \delta_{n,2}$, $\beta = 1$ and $k_\beta = 1$ one obtains

$$\begin{aligned} \psi_\beta^{(1)}(0, x; u_\beta^{(1)}) &= \frac{k_\beta}{\beta} \left(\varphi(0, x; 1 - \beta u_\beta^{(1)}) - 1 + \beta u_\beta^{(1)} \right) \\ &= \frac{k_\beta}{\beta} \left(\sum p_n (1 - \beta u_\beta^{(1)})^n - 1 + \beta u_\beta^{(1)} \right) \\ &= \frac{k_\beta}{\beta} \left(\beta^2 u_\beta^{(1)2} - \beta u_\beta^{(1)} \right) \\ &\rightarrow u^2 - u \end{aligned} \quad (26)$$

Therefore, in this case, one is led to the KPP equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - u^2 + u \quad (27)$$

However in this case, because $\beta = 1$ instead of $\beta \rightarrow 0$, the solution is given by $(1 - e^{-w})$ instead of (8). Because of the natural stochastic clock provided by the linear u term, a stochastic solution for the Cauchy problem of the KPP equation may be constructed by other method [4]. However, the interpretation as an exit measure, allows for the construction of solutions with arbitrary boundary conditions.

4 Superprocesses on signed measures and ultradistributions

Although the scaling limit $\beta \rightarrow 0$ of measure-valued superprocesses allows the construction of solutions for equations which do not possess a natural Poisson clock, it has the severe limitation of requiring a polynomial branching function $\varphi(s, x; z)$. This automatically restricts the nonlinear terms in the pde's to be powers of u . In addition, these terms must be such that all coefficients in the z^n expansion in Eq.(13) be positive to be interpretable as branching probabilities. As seen before, it was this requirement that led to the restriction $1 < \alpha \leq 2$ in (21).

The variable z that appears in $\varphi_\beta(s, x; z)$ is in fact $z = e^{-\beta w(\tau-s, \xi_s)} = P_{0,x} e^{-\langle \beta f, X \rangle}$. When restricting the superprocess to measures, the delta measure, at each branching point, may at most branch into other deltas (with positive coefficients) and therefore $\varphi(s, x; z)$ must be a sum of monomials in z . When one generalizes to \mathcal{U}'_0 ultradistributions of point support¹, changes of sign and transitions from deltas to their derivatives are allowed. In the end, the exponential $e^{-\langle \beta f, X \rangle}$ will be computed by evaluation of the function on the ultradistribution that reaches the boundary. To find out the equation that is represented by the process one needs to compute the $\psi_\beta(0, x; u_\beta)$ of Eq.(17) for the corresponding $\varphi(s, x; z)$ in the $\beta \rightarrow 0$ limit. Recalling that $\varphi(s, x; z) = \varphi_\beta(s, \xi_s; e^{-\beta w(\tau-s, \xi_s)})$ and $z = e^{-\beta w_\beta}$, one concludes that there are basically two new transitions at the branching points:

- 1) A change of sign in the point support ultradistribution

$$e^{\langle \beta f, \delta_x \rangle} = e^{\beta f(x)} \rightarrow e^{\langle \beta f, -\delta_x \rangle} = e^{-\beta f(x)} \quad (28)$$

which corresponds to

$$z \rightarrow \frac{1}{z} \quad (29)$$

and

¹Because distributions of point support are a finite sum of deltas and their derivatives [24], one could have considered only distributions of point support rather than compact support ultradistributions \mathcal{U}'_0 . However in \mathcal{U}'_0 one is not restricted to finite sums.

2) A change from $\delta^{(n)}$ to $\pm\delta^{(n+1)}$, for example

$$e^{\langle\beta f, \delta_x\rangle} = e^{\beta f(x)} \rightarrow e^{\langle\beta f, \pm\delta'_x\rangle} = e^{\mp\beta f'(x)} \quad (30)$$

which corresponds to

$$z \rightarrow e^{\mp\partial_x \log z} \quad (31)$$

Case 1) corresponds to an extension of superprocesses on measures to superprocesses on signed measures and the second to superprocesses in \mathcal{U}'_0 .

How these transformations provide stochastic representations of solutions for other classes of pde's, will be illustrated by two examples:

First, let

$$\varphi^{(1)}(0, x; z) = p_1 e^{\partial_x \log z} + p_2 e^{-\partial_x \log z} + p_3 z^2 \quad (32)$$

This branching function means that at the branching point, with probability p_1 a derivative is added to the propagating ultradistribution, with probability p_2 a derivative is added plus a change of sign and with probability p_3 the ultradistribution branches into two identical ones. Using the transformation and scaling limit (16) one has, for small β

$$z \rightarrow e^{\mp\partial_x \log z} = e^{\mp\partial_x \log(1 - \beta u_\beta^{(1)})} = 1 \pm \beta \partial_x u_\beta^{(1)} + \frac{\beta^2}{2} \left\{ \left(\partial_x u_\beta^{(1)} \right)^2 \pm \partial_x u_\beta^{(1)2} \right\} + O(\beta^3) \quad (33)$$

$$z \rightarrow z^2 = \left(1 - \beta u_\beta^{(1)} \right)^2 = 1 - 2\beta u_\beta^{(1)} + \beta^2 u_\beta^{(1)2} \quad (34)$$

Then, computing $\psi_\beta(0, x; u_\beta^{(1)})$ with $p_1 = p_2 = \frac{1}{4}$ and $p_3 = \frac{1}{2}$ one obtains

$$\begin{aligned} \psi_\beta^{(1)}(0, x; u_\beta^{(1)}) &= \frac{k_\beta}{\beta} \left(\varphi^{(1)}(0, x; z) - z \right) \\ &= \frac{k_\beta}{\beta} \left(\varphi^{(1)}(0, x; 1 - \beta u_\beta^{(1)}) - 1 + \beta u_\beta^{(1)} \right) \\ &= \frac{k_\beta}{\beta} \left(\frac{1}{8} \beta^2 \left(\partial_x u_\beta^{(1)} \right)^2 + \frac{1}{2} \beta^2 u_\beta^{(1)2} + O(\beta^3) \right) \end{aligned} \quad (35)$$

meaning that, with $k_\beta = \frac{4}{\beta}$, the superprocess provides, in the $\beta \rightarrow 0$ limit, a solution to the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - 2u^2 - \frac{1}{2} (\partial_x u)^2 \quad (36)$$

For the second example a different scaling limit will be used, namely

$$u_\beta^{(2)} = \frac{1}{2\beta} (e^{\beta w_\beta} - e^{-\beta w_\beta}) \quad ; \quad f_\beta^{(2)} = \frac{1}{2\beta} (e^{\beta f} - e^{-\beta f}) \quad (37)$$

Notice that, as before, $u_\beta^{(2)} \rightarrow w_\beta$ and $f_\beta^{(2)} \rightarrow f$ when $\beta \rightarrow 0$. In this case with $z = e^{\beta w_\beta}$ one has

$$\begin{aligned} z &= -2\beta u_\beta^{(2)} + 2\sqrt{\beta^2 u_\beta^{(2)2} + 1} \\ &= 2 - 2\beta u_\beta^{(2)} + \beta^2 u_\beta^{(2)2} + O(\beta^4) \end{aligned} \quad (38)$$

and

$$\begin{aligned}\frac{1}{z} &= 2\beta u_\beta^{(2)} + 2\sqrt{\beta^2 u_\beta^{(2)2} + 1} \\ &= 2 + 2\beta u_\beta^{(2)} + \beta^2 u_\beta^{(2)2} + O(\beta^4)\end{aligned}\quad (39)$$

For the integral equation, instead of (18), one has

$$u_\beta^{(2)}(0, x) + \Pi_{0,x} \int_0^\tau ds \psi_\beta^{(2)}(s, \xi_s; u_\beta^{(2)}) = \Pi_{0,x} f_\beta^{(2)}(\tau, \xi_\tau) \quad (40)$$

with

$$\psi_\beta^{(2)}(0, x; u_\beta^{(2)}) = k_\beta \left(\frac{1}{2\beta} \left(\varphi(0, x; z) - \varphi\left(0, x; \frac{1}{z}\right) \right) - u_\beta^{(2)} \right) \quad (41)$$

Let now

$$\varphi^{(2)}(0, x; z) = p_1 z^2 + p_2 \frac{1}{z} \quad (42)$$

This branching function means that with probability p_1 the ultradistribution branches into two identical ones and with probability p_2 it changes its sign. Therefore, in this case, one is simply extending the superprocess construction to signed measures. Using (38) and (39) one computes $\psi_\beta^{(2)}(0, x; u_\beta^{(2)})$ obtaining

$$\psi_\beta^{(2)}(0, x; u_\beta^{(2)}) = k_\beta \left\{ -p_1 8u_\beta^{(2)} \left(1 + \frac{1}{2}\beta^2 u_\beta^{(2)2} \right) + p_2 u_\beta^{(2)} - u_\beta^{(2)} + O(\beta^4) \right\} \quad (43)$$

and with $p_1 = \frac{1}{10}$; $p_2 = \frac{9}{10}$ and $k_\beta = \frac{5}{2\beta^2}$ one obtains in the in the $\beta \rightarrow 0$ limit

$$\psi_\beta^{(2)}(0, x; u_\beta^{(2)}) \rightarrow -u_\beta^{(2)3} \quad (44)$$

meaning that this superprocess provides a solution to the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + u^3 \quad (45)$$

In conclusion: Extending the superprocess construction to signed measures and ultradistributions, stochastic solutions are obtained for a much larger class of partial differential equations.

5 Appendix: Proof of the lemma

Let

$$u(x, t) = \Pi_{0,x} \left\{ e^{-kt} u(\xi_t, 0) + \int_0^t k e^{-ks} \Phi(\xi_s, t-s) ds \right\} \quad (46)$$

Then

$$\begin{aligned} \Pi_{0,x} \int_0^t k u(\xi_s, t-s) ds &= \Pi_{0,x} \left\{ \int_0^t k e^{-k(t-s)} u(\xi_{s+t-s}, 0) ds \right. \\ &\quad \left. + \int_0^t k ds \int_0^{t-s} k ds' e^{-ks'} \Phi(\xi_{s+s'}, t-s-s') \right\} \end{aligned} \quad (47)$$

Summing (46) and (47)

$$\begin{aligned} &u(x, t) + \Pi_{0,x} \int_0^t k u(\xi_s, t-s) ds \\ &= \Pi_{0,x} \left\{ \left(e^{-kt} + \int_0^t k e^{-k(t-s)} ds \right) u(\xi_t, 0) \right. \\ &\quad \left. + k \int_0^t e^{-ks} \Phi(\xi_s, t-s) ds + k \int_0^t ds \int_0^{t-s} k ds' e^{-ks'} \Phi(\xi_{s+s'}, t-s-s') ds' \right\} \end{aligned} \quad (48)$$

Changing variables in the last integral in (48) from (s, s') to $(s, \sigma = s + s')$ one obtains for the last term

$$k \int_0^t d\sigma \int_0^\sigma k ds e^{-k(\sigma-s)} \Phi(\xi_\sigma, t-\sigma) ds$$

and finally

$$\begin{aligned} &u(x, t) + \Pi_{0,x} k \int_0^t u(\xi_s, t-s) ds \\ &= \Pi_{0,x} \left\{ u(\xi_t, 0) + k \int_0^t \Phi(\xi_s, t-s) ds \right\} \end{aligned} \quad (49)$$

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